Exercise Set #5

"Discrete Mathematics" (2025)

Exercise 6 is to be submitted on Moodle before 23:59 on March 24th, 2025

E1. **(a)** Denote b_n the n-th Catalan number. Show that

$$b_n = \frac{1}{n+1}\binom{2n}{n}$$

by completing the remaining calculation from the lecture.

(b) Verify the formula for the number of binary trees with $0 \leq n \leq 3$ vertices by drawing all binary trees with three or fewer vertices.

E2. Show that the solution of each of the counting problems below is the same. Try to list the objects in each case for $n = 3$ or $n = 4$.

(a) Number of paths a drunk frog moving away from a wall can take in $2n$ steps given that it returns to the same place. In each step, the frog can either move one inch away from the wall or one inch towards the wall and it starts and ends at 0 distance from the wall. Each step of the frog is of equal distance and the frog cannot jump through the wall.

(b) The number of ways in which n coins of 1 franc and n coins of 2 franc can be distributed among $2n$ people standing in a coffee machine queue in the following way: Each person gets exactly one coin, and when they start buying a coffee in the order they are standing, the coffee machine never runs out of change. The coffee costs 1 franc and the machine has no coins inside to begin with.

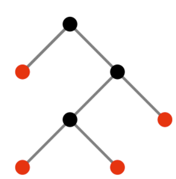
(c) The number of ways you can make arrange n pairs of parenthesis { "(", ")" } such that they form a valid expression. For example, (())() is a valid expression but ())() (is not a valid expression because the left and right-parentheses are not correctly matched.

(d) The number of ways a student can compute the product

$$A_1 \times A_2 \times A_3 \times \cdots \times A_{n+1}$$

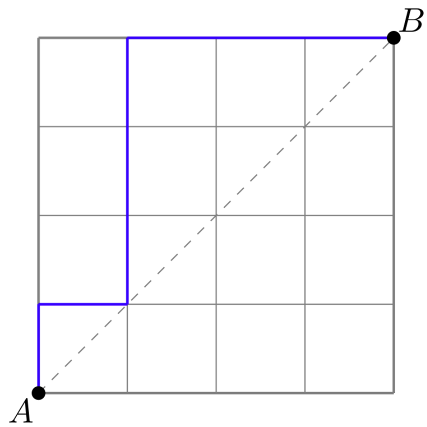
of $n+1$ square matrices $\{A_i\}_{i=1}^{n+1}$ of the same size if in each step he can multiply only two adjacent matrices.

(e) The number of "full" binary trees with $n+1$ leaves. A full binary tree is a binary tree in which either both the subtrees at each vertex are empty or both are non-empty. When both the subtrees are empty, such a vertex is called a leaf. Below is an example of such a tree with 4 leaves.



(f) The number of binary trees with n unlabelled vertices.

E3. Consider an $n \times n$ chessboard:



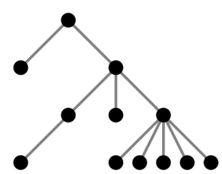
Consider the shortest paths from the corner A to the corner B following the edges of the squares (each of them consists of $2n$ edges).

(a) How many such paths are there?

(b) Show that the number of paths that never go below the diagonal (the line AB) is exactly b_n, i.e. the Catalan number. One such path is drawn in the figure.

E4. Let us denote the t_n to be the number of rooted plane trees with n vertices. They are defined as follows:

Each tree has a distinguished vertex called root along with a finite **ordered set** of trees which are called subtrees. For example, the following is a tree



(a) Take $t_0 = 0$. Show that the generating function $t(x) = \sum_{i=0}^{n} t_i x^i$ satisfies

$$t(x)(1 - t(x)) = x.$$

(b) Find t_n for $n = 1, 2, 3, 4, 5$. Do you recognize these numbers? Optional question: Why does this happen?

E5. Show that the number of ways to triangulate a regular polygon with $n+2$ labelled vertices is the Catalan numbers.

E6. (**Exercise to submit**)

(a) Using generating functions, show that $f_0 + f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$ where f_n is the n-th Fibonacci number.

(b) Consider a circle formed by $2n$ people. Let C_n be the number of ways in which n pairs can shake hands simultaneously **without** the handshakes crossing. Prove that C_n corresponds to the n-th Catalan number.